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DECOMPOSITION OF 3-CONNECTED GRAPHS

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Cunningham and Edmonds [4] have proved that a 2-connected graph G has a unique minimal decomposition into graphs, each of which is either 3-connected, a bond or a polygon. They define the notion of a good split, and first prove that G has a unique minimal decomposition into graphs, none of which has a good split, and second prove that the graphs that do not have a good split are precisely 3-connected graphs, bonds and polygons. This paper provides an analogue of the first result above for 3-connected graphs, and an analogue of the second for minimally 3-connected graphs. Following the basic strategy of Cunningham and Edmonds, an appropriate notion of good split is defined. The first main result is that if G is a 3-connected graph, then G has a unique minimal decomposition into graphs, none of which has a good split. The second main result is that the minimally 3-connected graphs that do not have a good split are precisely cyclically 4-connected graphs, twirls ($K_{3,n}$ for some $n \geq 3$) and wheels. From this it is shown that if G is a minimally 3-connected graph, then G has a unique minimal decomposition into graphs, each of which is either cyclically 4-connected, a twirl or a wheel.

1. Introduction

This paper describes a decomposition for 3-connected graphs. The decomposition is based on the 3-separations of a graph. The decomposition satisfies several properties, foremost of which is uniqueness. In the case that the graph is minimally 3-connected, then additional properties are satisfied.

Unique graph decompositions based on k-separations for $k = 1$ and 2 have been studied previously. Whitney [9] is perhaps to first to describe a unique graph decomposition that uses the notion of a k-separation. His decomposition is based on the 1-separations of a connected graph. Decomposition uniqueness results for 2-connected graphs based on 2-separations have been developed by MacLane [10], Tutte [18, Chapter 11], Hopcroft and Tarjan [8] and Cunningham and Edmonds [4]. Thus, the present work can be seen as a natural next step. (Robertson and Shih also have a unique decomposition for 3-connected graphs; this work is unpublished. Their decomposition is different from the one presented here in that the pieces of the decomposition are allowed to be structures that are more general than graphs.)

Of the works mentioned above, the one by Cunningham and Edmonds [4] was particularly influential in the development of this paper. Their paper contains a general decomposition theory. The decomposition of 3-connected graphs presented here does not fit into their theory, but many of the definitions and concepts intro-

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duced below are essentially those of Cunningham and Edmonds. The statements of Theorems 1.1–1.3 below are modeled closely after their work. In particular, Cunningham and Edmonds defined the notion of a good split (which is a special type of 2-separation) and first proved every 2-connected graph can be decomposed in a unique way into graphs none of which have a good split (cf. Theorem 1.2). Second, they characterized the graphs that do not have a good split as 3-connected graphs, bonds and polygons (cf. Theorem 1.3). Thus, they proved that a 2-connected graph can be decomposed in a unique way into graphs, each of which is either 3-connected, a bond or a polygon (cf. Theorem 1.1).

The definitions needed to give precise statements of these theorems are now supplied. Where G is a connected graph and F is a nonempty subset of $E(G)$, $G[F]$ denotes the subgraph of G having edge set F and having no isolated vertices, and $A(G,F)$ denotes the set $V(G[F]) \cap V(G[E(G)-F])$. For a positive integer k, a *k-separation* of G is defined to be a partition $\{E_1, E_2\}$ of $E(G)$ such that $|E_1| \geq k \leq |E_2|$ and $|A(G,E_1)| \leq k$. For a positive integer n, the graph G is *n-connected* if G has no k-separation for any $k < n$. A fundamental property, known as Menger's Theorem [13], of n-connected graphs having at least $n+1$ vertices is that every pair of vertices are joined by at least n internally disjoint paths. The graph G is *minimally n-connected* if it is n-connected and for every $e \in E(G)$, the graph $G \setminus e$ has an $(n-1)$-separation.

Let G be a 3-connected graph. Suppose $\{E_1, E_2\}$ is a 3-separation of G with $A(G,E_1) = \{x,y,z\}$. From each of $G[E_i]$, $i \in \{1,2\}$, a graph G_i is constructed as follows. First, let $\{e,f,g,t\}$ be a set disjoint from $E(G) \cup V(G)$. Add e, f, g and t to each of $G[E_i]$ so that t is a vertex that is adjacent precisely to x, y and z via edges e, f and g, respectively. Now contract those edges of $\{e,f,g\}$ that are incident to a vertex of degree two, with the new vertex of the resulting graph that is adjacent to x, y and z taking the name t. Finally, in the case that any of $\{e,f,g\}$ are contracted, a renaming procedure is carried out. In particular, suppose e (say) is contracted in the graph obtained from $G[E_1]$. Let p be the unique edge of E_1 adjacent to e before it is contracted. Then in the graph obtained from $G[E_2]$, rename the edge e to p. (Note, by 3-connectivity, the edge e is not contracted in the graph obtained from $G[E_2]$.) The resulting set $\{G_1, G_2\}$ is called the *simple decomposition* of G *associated with* the 3-separation $\{E_1, E_2\}$, the set of edges $E(G_1) \cap \{e,f,g\}$ and the vertex t. The set of edges $E(G_1) \cap \{e,f,g\}$ and the vertex t are the *marker* edges and *marker* vertex of the simple decomposition, respectively. An example is given in Figure 1; marker edges are dashed and marker vertices are darkened.

The following definitions are essentially taken from Cunningham and Edmonds [4]. A *decomposition* D of a 3-connected graph G is defined inductively to be either $\{G\}$ or a set obtained from a decomposition D' of G by replacing a member H of D' by the members of a simple decomposition of H. In the latter case, D is a *simple refinement* of D'. More generally, a decomposition D_t is a *refinement* of a decomposition D_1 if there exists a sequence $D_1, \ldots, D_t$ of decompositions such that for $1 \leq i \leq t-1$, D_{i+1} is a simple refinement of D_i. Note that a decomposition is a refinement of itself. If $t > 1$, then D_t is a *proper* refinement of D_1. Two decompositions D and D' of G are *equivalent* if D' can be obtained from D by replacing some of the marker edges and vertices of the members of D by marker edges and vertices of the members of D'. A decomposition D of G is *unique* with

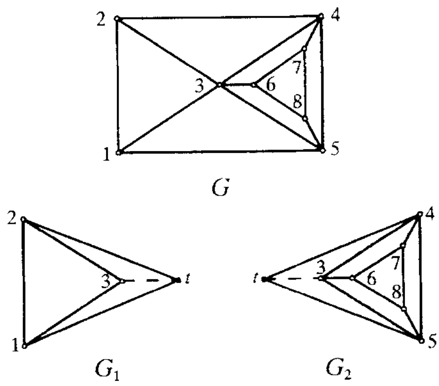

Fig. 1

respect to some property P if D satisfies P and any other decomposition of G that satisfies P is equivalent to D. A decomposition D is *minimal* with respect to some property P if D satisfies P but no decomposition having D as a proper refinement satisfies P.

For the most part, this paper is interested in unique decompositions, and therefore the choice of names for the marker edges and vertices will not be stressed. That is, a phrase such as *the simple decomposition of G associated with the 3-separation* $\{E_1, E_2\}$ will be used without reference to a particular set of marker edges or marker vertex.

A motivating question is: Does every 3-connected graph have a unique minimal decomposition each member of which is 4-connected? The answer is no, even for minimally 3-connected graphs; there exist many counterexamples. Of particular interest are the families of counterexamples given by twirls and wheels. A *twirl* is a complete bipartite graph having exactly three vertices in one member of its vertex partition and at least three in the other. A *wheel* is a graph obtained from a cycle with $n \geq 3$ edges by adding a new vertex and n new edges such that the new vertex is adjacent precisely to each vertex of the cycle.

Twirls and wheels are troublesome with respect to decompositions. First, note that if $\{E_1, E_2\}$ is a 3-separation of a twirl or wheel such that E_1 is a *triad* (the set of edges incident to a degree-three vertex), then G_2 of the associated simple decomposition is isomorphic to G. It follows that no decomposition of a twirl or wheel has every member 4-connected. One strategy to overcome the above difficulty with triads is to ignore those 3-separations that arise from them. This leads to the following definition. A 3-separation $\{E_1, E_2\}$ of a 3-connected graph is a *cyclic* 3-separation if neither E_1 nor E_2 is a triad. A 3-connected graph is *cyclically 4-connected* if it has no cyclic 3-separation.

Even when restricted to cyclic 3-separations, twirls and wheels can have nonunique decompositions. This can be seen as follows. Consider the graph $G := K_{3,4}$, and let $\{T_1, \ldots, T_4\}$ be the four triads of G. Then $\{T_1 \cup T_2, T_3 \cup T_4\}$ and $\{T_1 \cup T_3, T_2 \cup T_4\}$ are cyclic 3-separations of G. Observe that each of these 3-separations leads to a simple decomposition that consists of two copies of $K_{3,3}$.

However, these simple decompositions are not equivalent. Also observe that the graph $K_{3,3}$ is cyclically 4-connected. Thus, G does not have a unique minimal decomposition every member of which is cyclically 4-connected. A similar phenomenon occurs with wheels; the wheel on five vertices can be decomposed into two copies of the wheel on four vertices in two non-equivalent ways. Twirls and wheels play a critical role in the decomposition of minimally 3-connected graphs. (The reader familiar with the work of Cunningham and Edmonds [4] will note an analogy between twirls and wheels in the decomposition of 3-connected graphs and bonds and polygons in the decomposition of 2-connected graphs. The behavior of twirls is similar to that of bonds, and the behavior of wheels is similar to that of polygons.)

The first main result of this paper is the following.

Theorem 1.1. *Every minimally 3-connected graph has a unique minimal decomposition with the property that every member is either cyclically 4-connected, a twirl or a wheel.*

The proof of Theorem 1.1 is in two main parts. To describe these parts some definitions are needed. Let $\{E_1, E_2\}$ be a cyclic 3-separation of a 3-connected graph G. Let A_3 be the set of edges that are incident, in $G[E_1]$ or $G[E_2]$, to a degree-one vertex of either $G[E_1]$ or $G[E_2]$, and let $A_i = E_i - A_3$ for $i \in \{1,2\}$. Observe that if $\{C_1, C_2\}$ is a partition of A_3, then $\{A_1 \cup C_1, A_2 \cup C_2\}$ is a 3-separation of G. Moreover, any two such 3-separations give rise to the same simple decomposition of G. These observations motivate the following definition; the ordered triple $A = \{A_1, A_2; A_3\}$ is a *split* of G. The vertices in the set $V(G[A_1]) \cap V(G[A_2])$ together with the edges of A_3 are called the *connections* of A (or the *A-connections*). The set of A-connections is denoted by $C(A)$.

Note that different 3-separations can give rise to the same split. However, any two such 3-separations also give rise to the same simple decomposition. Thus, a split uniquely defines the simple decomposition, and therefore the phrase *the simple decomposition associated with the split* is well defined, and will be used hereafter.

Observe that the split $A = \{A_1, A_2; A_3\}$ is uniquely determined by A_1. Precisely, a set $A_1 \subseteq E(G)$ *induces* a split of G if and only if $G[A_1]$ has no degree-one vertices and $\{A_1, E(G) - A_1\}$ is a cyclic 3-separation of G. Splits $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ are *distinct* if $A_1 \neq B_1$ and $A_1 \neq B_2$.

The following observations can be made about a split $A = \{A_1, A_2; A_3\}$ of a 3-connected graph G. First, A has exactly three connections. Second, each A-connection that is an edge has one end in $V(G[A_1]) - V(G[A_2])$ and the other in $V(G[A_2]) - V(G[A_1])$. Finally, no two edges that are A-connections are adjacent. Using the first observation, the split A is said to be *type i* for $i \in \{0,1,2,3\}$, if exactly i of the A-connections are edges. The four types of splits are depicted in Figure 2.

Two splits $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ are said to *cross* if $A_i - B_j$ and $B_j - A_i$ are nonempty for all $i, j \in \{1,2\}$. Two distinct splits that do not cross are *compatible*. A *good* split is one that is not crossed by any other split. (Observe that the splits of $K_{3,4}$ discussed earlier cross.)

In Section 3, the following result, which is the second main result, is proved. Note it applies to all 3-connected graphs, not just those that are minimally 3-connected.

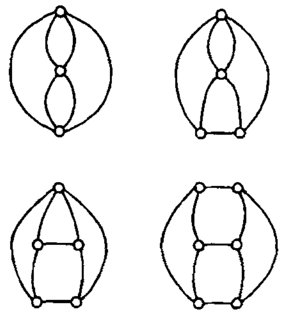

Fig. 2

Theorem 1.2. *Every 3-connected graph has a unique minimal decomposition with the property that no member has a good split.*

In Section 4, the third, and last, main result is proved.

Theorem 1.3. *A minimally 3-connected graph does not have a good split if and only if it is either cyclically 4-connected, a twirl or a wheel.*

The combination of Theorems 1.2 and 1.3 together with the fact (Lemma 2.5) that the members of a simple decomposition of a minimally 3-connected graph are minimally 3-connected evidently yields Theorem 1.1. A natural problem is to characterize those 3-connected graphs that do not have a good split. This problem seems difficult and is not solved here; the techniques developed in Section 4 do not lend themselves to its solution. However, the following is known. The class of 3-connected graphs that do not have a good split properly contains the class of minimally 3-connected graphs that do not have a good split. To see this, observe that a graph obtained from a wheel by joining a pair of non-adjacent degree-three vertices is a 3-connected graph that does not have a good split. Such a graph is neither cyclically 4-connected, a wheel or a twirl.

There are two keys to the proof of Theorem 1.2. Consider a pair of compatible splits. The first key is in showing that compatible splits are "hereditary". That is, the first split has a "descendant" in one of the graphs that is a member of the simple decomposition that results from the second split. This is done in Lemma 3.3. The second key is in showing that compatible splits "commute". That is, the decomposition obtained from the first split and the descendant of the second is equivalent to the one obtained from the second split and the descendant of the first. This is done in Lemma 3.7. The basic idea of the hereditary and commutativity properties are taken from Cunningham end Edmonds [4], but the proofs of Lemmas 3.3 and 3.7 are considerably more complicated than the analogous results for 2-connected graphs.

Theorem 1.3 is a structural result for minimally 3-connected graphs. Other such results for minimally 3-connected graphs are found in the work of Halin [6,7], Mader [11,12] and Dawes [5]. Minimally 3-connected graphs include Halin graphs and 3-connected cubic graphs. Some extensions to 4-connectivity can be found in Robertson [15].

The notion of the simple decomposition of a 3-connected graph that is introduced here is not new. For example, it can be seen in the work of Cornuéjols, Naddef and Pulleyblank [3] on Halin graphs, and is related to the matroid work of Rajan [14], Seymour [16] and Truemper [17]. However, the fact that it leads to a unique decomposition is new.

The organization of the paper is as follows. The next section establishes some basic properties of simple decompositions and of splits. Section 3 contains the proof of Theorem 1.2. Section 4 contains the proof of Theorem 1.3. Section 5 discusses an algorithm for computing the decomposition specified in Theorem 1.1. Section 6 closes the paper with a discussion of some related topics.

Undefined notation and terminology is for the most part consistent with Bondy and Murty [2]. The symbol $\subseteq$ denotes containment, whereas $\subset$ denotes proper containment.

2. Properties of Simple Decompositions and Splits

The results of this section establish basic properties of the members of simple decompositions and of splits.

Lemma 2.1. *Let G be a 3-connected graph having a split $\{A_1,A_2;A_3\}$. Then $G[A_1]$ is 2-connected.*

Proof. Suppose $G[A_1]$ has a 1-separation $\{E_1,E_2\}$. Observe that $G[A_1]$ has no vertices of degree one, since the definition of the split implies that any such vertex would have degree at most two in G. It follows that $|E_i| \geq 2$ for $i \in \{1,2\}$. If $A(G,A_1)\cap V(G[E_1])$ contains at most one vertex, then $|A(G,E_1)| \leq 2$, implying that G has a 2-separation. Thus, $|A(G,A_1)\cap V(G[E_1])| \geq 2$. Similarly $|A(G,A_1)\cap V(G[E_2])| \geq 2$, implying $|A(G,E_1)|=2$, a contradiction. ∎

Let $A=\{A_1,A_2;A_3\}$ be a split and let $\{G_1,G_2\}$ be the simple decomposition associated with A. Where t is the marker vertex of G_1, observe $G[A_1]=G_1\setminus t$.

Lemma 2.2. *Let G be a 3-connected graph. Then the members of the simple decomposition associated with a split of G are 3-connected.*

Proof. Let $\{G_1,G_2\}$ be the simple decomposition associated with a split $\{A_1,A_2;A_3\}$ of G, and let t be the marker vertex of G_1.

Suppose that G_1 has a 2-separation $\{E_1,E_2\}$. If $t\in A(G_1,E_1)$, then $G\setminus t$ has a 1-separation, contradicting Lemma 2.1. Thus, $t \notin A(G_1,E_1)$. Without loss of generality, assume that $t\in V(G_1[E_2])$. Then $E_1\subseteq A_1$ implying that $A(G,E_1)=2$. That is, $\{E_1,E(G)-E_1\}$ is a 2-separation of G, a contradiction.

Finally, suppose that G_1 has a 1-separation that is not a 2-separation. In this case G_1 must have either a loop or a vertex of degree one; each of these possibilities is easily shown to be impossible. ∎

The rest of this section deals with properties of splits of minimally 3-connected graphs. These properties are needed in Section 4.

A *triangle* is the edge set of a cycle having exactly three edges. The following result follows easily from Tutte [18,10.54].

Lemma 2.3. *Let G be a minimally 3-connected graph with at least four vertices. Then every triangle of G has nonempty intersection with two distinct triads of G.* ∎

Lemma 2.4. *Let G be a minimally 3-connected graph having a split $A = \{A_1, A_2; A_3\}$. Then no edge of G joins two vertices that are A-connections.*

Proof. Let e be an edge of G that joins two vertices that are A-connections, say p and q. Since G is minimally 3-connected, $G\backslash e$ has a 2-separation $\{E_1, E_2\}$. Without loss of generality, p is incident only to members of E_1. Since p is incident to edges from both A_1 and A_2, it follows that $E_1 \cap A_i$ is nonempty for $i \in \{1,2\}$. Likewise, $E_2 \cap A_i$ is nonempty for $i \in \{1,2\}$. Thus, $V(G[A_1])$ and $V(G[A_2])$ each contain at least one member from $A(G\backslash e, E_1)$, implying $\{A_1 \cap E_1, A_1 \cap E_2\}$ is a 1-separation of $G[A_1] \backslash e$. Therefore, by Lemma 2.1, $e \in A_1$, Now if $V(G[A_2])$ contains just one member of $A(G\backslash e, E_1)$, then $G[A_2]$ has a 1-separation, contradicting Lemma 2.1. Thus, $V(G[A_2])$ contains two members of $A(G\backslash e, E_1)$ implying that one of these members is the unique vertex r (say) in $A(G, A_1) - \{p,q\}$. It now follows that $A(G, A_1 \cap E_1)$ is the set consisting of p (say) and r, and $A(G, A_1 \cap E_2)$ is the set consisting of q and r. Since G is 3-connected, it must be that $A_1 \cap E_1$ and $A_1 \cap E_2$ each consist of a single edge, say f and g, respectively. Now $A_1 = \{e, f, g\}$ is a triangle of G. By Lemma 2.3, A_1 has nonempty intersection with two distinct triads of G, implying that $G \backslash A_1$ has two degree-one vertices. But then at most one A-connection can be a vertex, a contradiction. ∎

Lemma 2.5. *Let G be a minimally 3-connected graph. Then the members of the simple decomposition associated with a split of G are minimally 3-connected.*

Proof. Let $\{G_1, G_2\}$ be the simple decomposition associated with a split $\{A_1, A_2; A_3\}$ of G, and let t be the marker vertex of G_1. By Lemma 2.2, the graph G_1 is 3-connected. Thus, it suffices to show that $G_1 \backslash e$ has a 2-separation for each $e \in E(G_1)$. Let $e = uv \in E(G_1)$.

If e is incident to t, then $G_1 \backslash e$ is easily seen to have a 2-separation as follows. Let E_1 be the set consisting of the two edges, other than e, incident to t and let $E_2 = E(G_1) - E_1$. Then $\{E_1, E_2\}$ is a 2-separation of G_1.

Now suppose that e is not incident to t. Then $e \in E(G)$, and so $G \backslash e$ has a 2-separation $\{F_1, F_2\}$. Note that a collection of internally disjoint (u,v)-paths in G contains at most three paths. Moreover, given a collection of three such paths, one path consists just of the edge e, and the remaining two paths must each use exactly one of the vertices in $A(G \backslash e, F_1)$.

Suppose $A(G \backslash e, F_1) \cap V(G[A_1]) = \emptyset$. Then each (u,v)-path in $G\backslash e$ contains a vertex of $V(G[A_2]) - V(G[A_1])$. Therefore, there cannot exist two internally disjoint (u,v)-paths in $G[A_1]$, contradicting Lemma 2.1.

Suppose $A(G \backslash e, F_1) \cap V(G[A_1])$ contains exactly one vertex. Then there does not exist two internally disjoint (u,v)-paths in $G[A_1]\backslash e$, implying that any collection of internally disjoint (u,v)-paths of $G_1 \backslash e$ contains at most two paths. It follows that $G_1 \backslash e$ has a 2-separation.

Finally, suppose $A(G \backslash e, F_1) \subseteq V(G[A_1])$. Suppose that neither F_1 nor F_2 is contained in A_1. Then by Lemma 2.4, there exist distinct vertices $p \in V(G[F_1]) - V(G[A_1])$ and $q \in V(G[F_2]) - V(G[A_1])$. Since each (p,q)-path in G must contain either the edge e or a vertex in $A(G\backslash e, F_1)$, the existence of three internally disjoint

(p,q)-paths in G is impossible, a contradiction. Therefore F_1 (say) is contained in A_1. It follows that $\{F_1, E(G_1 \setminus e) - F_1\}$ is a 2-separation of $G_1 \setminus e$. ∎

Lemma 2.6. *Let $A = \{A_1, A_2; A_3\}$ be a split of a minimally 3-connected graph. Then $|V(G[A_1]) - V(G[A_2])| \geq 2$.*

Proof. The result is true if A is type 2 or 3 because each edge that is an A-connection has an end in the set $V(G[A_1]) - V(G[A_2])$.

If A is type 0 or 1, the there exists a vertex v in $V(G[A_1]) \cap V(G[A_2])$. By the definition of a split, v has degree at least two in $G[A_1]$, implying, by Lemma 2.4, that v is adjacent to at least two vertices of $V(G[A_1]) - V(G[A_2])$. ∎

3. Uniqueness of the Decomposition.

The main result of this section is that every 3-connected graph has a unique minimal decomposition no member of which has a good split.

The next lemma shows a certain symmetry holds between a pair of compatible splits.

Lemma 3.1. *Let $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ be splits of a 3-connected graph G with $B_1 \subset A_1$. Then $A_2 \subset B_2$.*

Proof. Since $B_1 \subset A_1$, it follows that $A_2 \cup A_3 \subset B_2 \cup B_3$. If A_2 is not a proper subset of B_2, then, since A and B are distinct, $A_2 \cap B_3$ is nonempty. However, each edge of B_3 is incident to a degree-one vertex of $G \setminus B_1$. Since $B_1 \subset A_1$, it follows that some edge of $G[A_2]$ is incident to a degree-one vertex, contradicting Lemma 2.1. ∎

Lemma 3.2. *Let $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ be crossing splits of a 3-connected graph G. Then $A_1 \cap B_1$ is nonempty.*

Proof. Suppose that $A_1 \cap B_1$ is empty. Then $A_1 \subseteq B_2 \cup B_3$ and $B_1 \subseteq A_2 \cup A_3$. Since A and B cross, it follows that $A_1 \cap B_3$ is nonempty. Thus, there exists an edge of $A_1 \cap B_3$ that is incident to a degree-one vertex of $G \setminus B_1$, which implies that $G[A_1]$ has a degree-one vertex, contradicting Lemma 2.1. ∎

The following lemma establishes a hereditary property for compatible splits.

Lemma 3.3. *Let $A = \{A_1, A_2; A_3\}$ be a split of a 3-connected graph G, and let $B_1 \subset A_1$. Let $\{G_1, G_2\}$ be the simple decomposition associated with A. Then B_1 induces a split of G if and only if it induces a split of G_1. Moreover, if B_1 induces a good split of G, then it induces a good split of G_1, and if A is a good split of G and B_1 induces a good split of G_1, then B_1 induces a good split of G.*

Proof. From the definition of the simple decomposition it follows that $A(G, B_1) = A(G_1, B_1)$.

Now suppose that B_1 induces a split of G. By the definition of a simple decomposition, $|E(G_1) - A_1| = 3$. Since $B_1 \subset A_1$, it follows that $|E(G_1) - B_1| \geq 4$, and thus $E(G_1) - B_1$ is not triad of G_1. Since B_1 induces a split of G, it follows that $|B_1| \geq 3$ and that B_1 is not a triad of G_1. Therefore $\{B_1, E(G_1) - B_1\}$ is a cyclic 3-separation of G_1, implying that B_1 induces a split of G_1.

Now suppose that B_1 induces a split of G_1. Then $|B_1| \geq 3$ and B_1 is not a triad of G. Since $B_1 \subset A_1$, it follows that $A_2 \cup A_3 \subset E(G) - B_1$. Therefore $|E(G) - B_1| \geq 4$, and hence $E(G) - B_1$ is not triad. Thus, $\{B_1, E(G) - B_1\}$ is a cyclic 3-separation of G, implying that B_1 induces a split of G.

Let $B = \{B_1, B_2; B_3\}$ and $B' = \{B'_1, B'_2; B'_3\}$ be the splits of G and G_1, respectively, induced by B_1, with $B_1 = B'_1$.

Now suppose that B is a good split of G but that B' is crossed by another split $X = \{X_1, X_2; X_3\}$. The first step is to show that either X_1 or X_2 is contained in A_1. Let $\{e, f, g\}$ be the set of edges incident to the marker vertex of G_1. Then $\{e, f, g\} = E(G_1) - A_1$. If neither X_1 nor X_2 is contained in A_1, then both $X_1 \cap \{e, f, g\}$ and $X_2 \cap \{e, f, g\}$ are nonempty. Moreover, from the definition of a split, it follows that X_1 (say) contains exactly two edges from $\{e, f, g\}$, say e and f. But then g is incident to a degree-one vertex in $G_1 \backslash X_1$, implying that $g \in X_3$, a contradiction. Thus, assume that $X_1 \subset A_1$. Now by the above, X_1 induces a split $Y = \{Y_1, Y_2; Y_3\}$ of G with $Y_1 = X_1$.

Since B and Y are compatible splits of G, Lemma 3.1 implies either $B_1 \subset Y_1$, $Y_1 \subset B_1$, $B_1 \subset Y_2$, or $Y_2 \subset B_1$. Since B' and X are crossing splits of G_1 and $B_1 = B'_1$ and $Y_1 = X_1$, it follows that $B_1 - Y_1$ and $Y_1 - B_1$ are nonempty. Thus, either $B_1 \subset Y_2$ or $Y_2 \subset B_1$.

If $B_1 \subset Y_2$, then $B'_1 \subset Y_2$, which implies $B'_1 \cap Y_1$ is empty. Therefore, $B'_1 \cap X_1$ is empty, contradicting Lemma 3.2. If $Y_2 \subset B_1$, then $Y_2 \subset A_1$, since $B_1 \subset A_1$. Lemma 3.1 applied to A and Y implies that $A_2 \subset Y_1$, and thus $A_2 \subset X_1$. But $X_1 \subset A_1$ leads to the contradiction that $A_2 \subset A_1$.

Finally suppose that B' is a good split of G_1, but that B is not a good split of G. Also suppose that A is a good split of G. Let $Y = \{Y_1, Y_2; Y_3\}$ be a split of G that crosses B. Since A is a good split of G, by Lemma 3.1 and by interchanging Y_1 and Y_2, if necessary, it can be assumed that either $Y_1 \subset A_1$ or $Y_1 \subset A_2$. In the latter case, $B_1 \cap Y_1$ is empty, contradicting Lemma 3.2. Thus, $Y_1 \subset A_1$. Therefore, by the above, Y_1 induces a split $X = \{X_1, X_2; X_3\}$ of G_1, with $X_1 = Y_1$.

Since B' is a good split, either $B'_1 \subset X_1, B'_2 \subset X_2$, $B'_1 \subset X_2$, or $B'_2 \subset X_1$. The first case can be ruled out since $B'_1 = B_1$, $X_1 = Y_1$ and the fact that B and Y are crossing splits of G. The second case is similarly ruled out by first applying Lemma 3.1. If $B'_1 \subset X_2$, then $B_1 \subset X_2$, which implies $B_1 \cap X_1$ is empty. Therefore $B_1 \cap Y_1$ is empty, contradicting Lemma 3.2. If $B'_2 \subset X_1$, then $B'_2 \subset A_1$, since $X_1 = Y_1 \subset A_1$. But B'_2 contains the set of edges incident to the marker vertex of G_1. A contradiction is obtained since these edges are not in A_1. ∎

Let $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ be splits of a 3-connected graph G such that $B_1 \subset A_1$. Where $\{G_1, G_2\}$ is the simple decomposition of G associated with A, the split of G_1 induced by B_1 is called the *descendant* of B in $\{G_1, G_2\}$. More generally and inductively, let D be a decomposition of G and $B' = \{B'_1, B'_2; B'_3\}$ be the descendant of B in D. Then for any simple refinement D' of D having a member H such that B'_1 (say) is contained in $E(H)$, the split of H induced by B'_1 is the *descendant* of B in D'.

A split is said to *generate* its associated simple decomposition. More generally and inductively, if D is a decomposition generated by a sequence $A^1, \ldots, A^{t-1}$ of pairwise-compatible splits of G, and A^t is a split of G that is compatible with each

$A^i, 1 \le i \le t-1$, then the decomposition D' *generated* by the sequence $A^1, \ldots, A^t$ is the decomposition obtained from D by replacing the (unique) member of D that has the descendant of A^t as a split by the members of the simple decomposition associated with this split.

It might seem plausible that some reordering of the sequence $A^1, \ldots, A^t$ can generate a decomposition different from D. Most of the remainder of this section is devoted to showing that this is not the case. That is, any ordering of a set of pairwise-compatible splits of G generates the same decomposition. The key fact is that any pair of compatible splits commute.

Lemma 3.4. *Let $A = \{A_1, A_2; A_3\}$ be a split of a 3-connected graph G, and let $B_1 \subset A_1$ induce a split B of G. Let $\{G_1, G_2\}$ be the simple decomposition associated with A, and let B' be the split of G_1 induced by B_1. Then either $C(B') = C(B)$ or $C(B') = (C(B) - \{v\}) \cup \{e\}$, where v is a vertex connection of A and e is the marker edge of G_1 incident to v.*

Proof. Consider an edge e that is a B-connection. Then there exists a vertex v that is an end of e and has degree one in $G \setminus B_1$. Suppose that $e \in A_1$. Then $e \in E(G_1)$. If e is not a B'-connection, then v is not a degree-one vertex of $G_1 \setminus B_1$, which implies that $v \in A(G, A_1)$. But then v is incident in G to at least one edge not in A_1, and thus v does not have degree one in $G \setminus B_1$, a contradiction. Thus, if $e \in A_1$, then e is a B'-connection. Now suppose that $e \notin A_1$. Then v has degree one in $G \setminus A_1$, and so e is an A-connection. Thus, $e \in E(G_1)$. Moreover, v is incident to exactly the same edges in G as in G_1, and so e is a B'-connection.

Now consider a vertex v that is a B-connection. Suppose that v is not a B'-connection. Since $A(G, B_1) = A(G_1, B_1)$, it must be that v is the end of an edge e that is a B'-connection. Since $e \in E(G_1)$, either $e \in (A_1 \cup A_3) - B_1$ or e is a marker edge of G_1. Since v is a B-connection, it is incident to at least two edges not in B_1. Since in G_1, the vertex v is incident to only one edge not in B_1, namely e, it must be that $v \in A(G, A_1)$. If $e \in A_1 - B_1$, then, in G_1, the vertex v is incident to at least two edges not in B_1, namely e and the edge joining v to the marker vertex of G_1, a contradiction. If $e \in A_3 - B_1$, then v has degree one in $G \setminus B_1$, contradicting the fact that v is a B-connection. Thus, e is a marker edge.

The final part is to show that there is at most one vertex that is a B-connection, but not a B'-connection. Suppose that there exist two such vertices, say u and v. Then by the above paragraph, there exist two marker edges of G_1, (the two such edges incident to u and v) that are B'-connections. This is a contradiction since no two edges that are connections can be adjacent. ∎

Lemma 3.5. *Let $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ be splits of a 3-connected graph G with $B_1 \subset A_1$. Let $\{G_1, G_2\}$ and $\{H_1, H_2\}$ be the associated simple decompositions of G, respectively. Let A' be the split of H_2 induced by A_2 and let B' be the split of G_1 induced by B_1. Then $C(B') = (C(B) - \{v\}) \cup \{e\}$ if and only if $C(A') = (C(A) - \{v\}) \cup \{f\}$, where v is an A-connection and a B-connection, and e is the marker edge of G_1 incident to v and f is the marker edge of H_2 incident to v.*

Proof. Suppose $C(B') = (C(B) - \{v\}) \cup \{e\}$, where v is an A-connection, and e is the marker edge of G_1 that is incident to v. If v is an A'-connection, then in H_2, the vertex v is incident to at least two edges of $E(H_2) - A_2$. At least one of the edges of $E(H_2) - A_2$ incident to v in H_2 must be in A_1. Since v is a B-connection, but not a B'-connection, every edge of A_1 incident to v is also an edge of B_1. This is a contradiction since H_2 has no edges from B_1. Therefore by Lemma 3.4, $C(A) = (C(A') - \{v\}) \cup \{f\}$, where f is the marker edge of H_2 incident to v. ∎

Lemma 3.6. *Let $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ be splits of a 3-connected graph G such that $B_1 \subset A_1$. Let $\{K_1, K_2, K_3\}$ be the decomposition of G generated by the sequence A, B with $B_1 \subset E(K_1)$ and $A_2 \subset E(K_2)$. Then K_1 and K_2 each contain exactly one marker vertex, say t_1 and t_2, respectively. Furthermore, K_3 contains exactly two marker vertices, and these two vertices are t_1 and t_2.*

Proof. Let $\{G_1, G_2\}$ be the simple decomposition of G associated with A, and let t_2 be the marker vertex. Then t_2 is the only marker vertex of G_2. Moreover $K_2 = G_2$. Let B' be the split of G_1 induced by B_1, and let $\{K_1, K_3\}$ be the split of G_1 associated with B'. Let t_1 be the associated marker vertex. Then t_1 is a vertex of K_1. It remains to show that t_2 is not a vertex of K_1 and that t_2 is a vertex of K_3.

First, observe that in G_1, the vertex t_2 is not incident to any edge of A_1, and therefore it is not incident to any edge of B_1. From this it follows directly that t_2 is not a vertex of K_1. Also, it follows that t_2 has degree greater than one in $G_1 \setminus B_1$, and therefore t_2 is a vertex of K_3. ∎

The following lemma establishes a commutativity property for compatible splits.

Lemma 3.7. *Let $A = \{A_1, A_2; A_3\}$ and $B = \{B_1, B_2; B_3\}$ be splits of a 3-connected graph G such that $B_1 \subset A_1$. Then the decomposition of G generated by the sequence A, B is equivalent to the decomposition generated by the sequence B, A.*

Proof. Let $\{K_1, K_2, K_3\}$ and $\{J_1, J_2, J_3\}$ be the decompositions of G generated by the sequences A, B and B, A, respectively. Moreover, assume that B_1 is a subset of $E(K_1)$ and $E(J_1)$, and that A_2 is a subset of $E(K_2)$ and $E(J_2)$.

By Lemma 3.6, K_1 and K_2 each contain exactly one marker vertex, say t_1 and t_2, respectively. Moreover, t_1 and t_2 are distinct vertices in K_3. By renaming, assume that the marker vertex that is in J_1 and J_3 is called t_1 and that the marker vertex that is in J_2 and J_3 is called t_2. Using symmetry, the equivalence between the decompositions will follow by verifying two facts. First, if a given pair of vertices in K_i, for some $i \in \{1,2,3\}$, are joined by an edge $e \in E(G)$, then the same pair of vertices are joined by e in J_i, and second, if a given pair of vertices are joined by a marker edge in K_i, for some $i \in \{1,2,3\}$, then the same pair of vertices are joined by a marker edge in J_i.

Consider an edge e of G that is incident to a non-marker vertex u in K_i, for some $i \in \{1,2,3\}$. From the definition of simple decomposition, it follows that $E(G) \cap E(K_i) = E(G) \cap E(J_i)$. It also follows that e is incident to u in G, and that there exists a $j \in \{1,2,3\}$ such that e is incident to u in J_j. Suppose that $i \neq j$. Then either i or j is in $\{1,2\}$. Suppose the $i = 1$; the other cases are analogous. Since e is in J_j, and thus in K_j, it follows that e is a B'-connection. By Lemma 3.4, it is also a B-connection. In K_1, the vertex u is incident only to edges from B_1 and to e. Moreover, since e is a B-connection, it follows from the definition of the simple decomposition that in G, the vertex u is incident only to edges from B_1 and to e. Again, from the definition of simple decomposition, in J_j, the only edges of

G that are incident to u are edges from B_1 and e. But then, it must be that $j=1$, a contradiction.

Consider an edge e of G that is incident to t_1 in K_1 and K_3. Then e is a B'-connection, and by Lemma 3.4, the edge e is a B-connection. Thus, e is incident to t_1 in J_1 and J_3. Now consider an edge e that is incident to t_2 in K_2 and K_3. Then e is an A-connection, and therefore by Lemma 3.4, the edge e is an A'-connection. Thus, e is incident to t_2 in J_2 and J_3.

Consider the case of a marker edge e, which appears in at least one of K_1, K_2 or K_3. Observe that e must appear in K_3 and at least one of K_1 or K_2.

First, consider the case when e is in K_1 and K_3, but not K_2. Let the ends of e in K_1 be u and t_1. Evidently, e also has ends u and t_1 in K_3. This implies that u is a B'-connection, and by Lemma 3.4, the vertex u is also a B-connection. Since u is a B-connection, there exists a marker edge f that in J_1 has ends u and t_1. The edge f also is in J_3, and in J_3 it is incident to t_1. It remains to verify that the other end of f in J_3 is indeed u. Denote by v the end of f in J_3 that is not equal to t_1. If $u \neq v$, then it must be that $v=t_2$. If $v=t_2$, then it must be that f is an A'-connection. But then Lemma 3.4 implies that u is an A-connection. A contradiction is now achieved since Lemma 3.5 implies that u is not a B'-connection.

Second, consider the case when e is in K_2 and K_3, but not K_1. Let the ends of e in K_2 be u and t_2. Since e is not in K_1, the ends of e in K_3 are also u and t_2. This implies that u is an A-connection. If u is an A'-connection, then it follows that a marker edge f joins u and t_2 in both J_2 and J_3. On the other hand, suppose that u is not an A'-connection. Then by Lemma 3.4, the vertex u is a B-connection. By Lemma 3.5, u is not a B'-connection. Moreover, Lemma 3.5 implies that e is a B'-connection, implying that e is in K_1, a contradiction.

Finally, consider the case when e is in K_1, K_2 and K_3. Let the ends of e be u and t_1 in K_1, v and t_2 in K_2, and t_1 and t_2 in K_3. Since e is in K_1 and K_3, it must be the case that e is a B'-connection. Lemma 3.4 implies that there exists a vertex x of G such that $C(B')=(C(B)-\{x\})\cup\{e\}$. Moreover, x is an A-connection and a B-connection. Since x is an A-connection and e is a B'-connection, it follows that $u=v=x$. Now Lemma 3.5 implies that there exists an edge f that is an A'-connection such that $C(A')=(C(A)-\{x\})\cup\{f\}$. Moreover, since x is a B-connection and f is an A'-connection, it follows that f has ends x and t_1 in J_1, x and t_2 in J_2, and t_1 and t_2 in J_3. ∎

The next result shows that any set of pairwise-compatible splits of a 3-connected graph generates a unique decomposition. That is, the decomposition is independent of the order of the splits. This proof is taken from that of an analogous theorem of Cunningham and Edmonds [4].

Theorem 3.8. *Let G be a 3-connected graph. Then any set of pairwise-compatible splits generates a unique decomposition.*

Proof. Let $\{A^1,\ldots,A^t\}$ be a set of pairwise-compatible splits of G. Let $I=\{1,\ldots,t\}$ be the index set of the splits. Then for any fixed ordering of the set I, the decomposition generated is unique. Moreover, any fixed ordering of I can be obtained from any other one by a sequence of interchanges of adjacent elements. Thus, it suffices to show that the decomposition D generated by the ordering $J=1,2,\ldots,j-1,j,j+1,j+2,\ldots,t$ of I is the same as the decomposition D' generated by the ordering $J' = 1,2,\ldots,j-1,j+1,j,j+2,\ldots,t$. Let $D_0 = D_0' = \{G\}$ and let D_m (respectively D_m'), for $1 \le m \le k$, denote the decomposition of G generated by the set consisting of those splits having their index in the first m terms of J (respectively J'), and in the order specified by J (respectively J'). Clearly, $D_{j-1} = D_{j-1}'$. If the descendants of A^j and A^{j+1} are in different members of D_{j-1}, then evidently $D_{j+1} = D_{j+1}'$. If the descendants of A^j and A^{j+1} are in the same member of D_{j-1}, then it follows from Lemma 3.7 that $D_{j+1} = D_{j+1}'$. In either case, $D = D'$. ∎

The proof of Theorem 1.2 is now at hand. Again, it is based on the proof of an analogous theorem of Cunningham and Edmonds [4].

Proof of Theorem 1.2. Let G be a 3-connected graph, and let D be a decomposition of G that is generated by a set S of splits of G. If there is a good split of G that is not in S, then by Lemma 3.3 there exists a member of D having a good split. Therefore if no member of D has a good split, then it follows that every good split of G is in S. If every good split of G is in S, then D must be a refinement of the unique (by Theorem 3.8) decomposition D' of G generated by the set of good splits of G. By Lemma 3.3, no member of D' has a good split. The theorem now follows. ∎

4. Twirls, Wheels and Crossing Splits

The purpose of this section is to prove that a minimally 3-connected graph does not have a good split if and only if it is either cyclically 4-connected, a twirl or a wheel. Clearly, cyclically 4-connected graphs do not have good splits, since they do not have any splits.

Let $A = \{A_1, A_2; A_3\}$ be a split of a minimally 3-connected graph G. Denote by S_i the set of vertices of $G[A_i]$, for $i \in \{1,2\}$. For a subset R of vertices of G, denote by $G[R]$ the subgraph of G induced by R. By Lemma 2.4, $G[S_1] = G[A_1]$ and $G[S_2] = G[A_2]$. Thus, given the graph G, the split A is uniquely determined by $S = \{S_1, S_2\}$. For the arguments in this section, it will be easier to deal with S rather than A, and therefore the set $S = \{S_1, S_2\}$ is defined to be a *split*. Note that no confusion can arise between the two uses of the word *split* since the "edge" split A consists of three sets whereas the "vertex" split S consists of two sets.

Let $A = \{A_1, A_2; A_3\}$ and $S = \{S_1, S_2\}$ be splits of a minimally 3-connected graph G such that $G[A_i] = G[S_i]$ for $i \in \{1,2\}$. The *connections* of S (or the *S-connections*) are defined to be the A-connections. Note a vertex of G is an S-connection if and only if it is in $S_1 \cap S_2$ and an edge of G is an S-connection if and only if it has one end in $S_1 - S_2$ and the other in $S_2 - S_1$. In addition, let $B = \{B_1, B_2; B_3\}$ and $T = \{T_1, T_2\}$ be splits of G such that $G[B_i] = G[T_i]$ for $i \in \{1,2\}$. Then S and T *cross* if A and B cross, or equivalently, by Lemma 2.4, if $S_i - T_j$ and $T_j - S_i$ are nonempty for all $i, j \in \{1,2\}$. The split S is *good* if it is not crossed by any other split. Finally, S is *type* i, for $i \in \{0,1,2,3\}$, if exactly i of the S-connections are edges.

The next two propositions show that twirls and wheels do not have good splits. In what follows, for a set R of vertices of a graph G, the set $A(G,E(G[R]))$ is abbreviated to $A(R)$.

Proposition 4.1. *If G is a twirl, then G does not have a good split.*

Proof. Let G be a twirl, and suppose $S=\{S_1,S_2\}$ is a split of G. Let $\{x,y,z\}$ be such that every edge of G joins a vertex in $\{x,y,z\}$ to a vertex not in $\{x,y,z\}$.

First, suppose that $\{x,y,z\} - (A(S_1) \cup A(S_2))$ is nonempty. Without loss of generality, suppose $x \in S_1 - A(S_1)$. Since x is adjacent to every vertex of $V(G) - \{x,y,z\}$, it follows that $V(G) - \{y,z\} \subseteq S_1$. By Lemma 2.6, $|S_2 - S_1| \geq 2$, and therefore $\{y,z\} = S_2 - S_1$. It follows from the definition of a split that no two members of $S_2 - S_1$ can be adjacent to the same member of $S_1 - S_2$. Thus, $S_1 - S_2 = \{x\}$, contradicting Lemma 2.6.

Second, suppose $\{x,y,z\} \subset A(S_1) \cup A(S_2)$. Then $|A(S_1) \cup A(S_2)| \geq 4$, which implies that some S-connection is an edge, say e. Without loss of generality, $e = ux$, where $u \in S_1 - \{x,y,z\}$. Note $x \in S_2 - S_1$. Since u is adjacent to y and z, it follows that $y,z \in S_1$. By Lemma 2.6, there exists another vertex, besides u, in $S_1 - S_2$. Since u is the only vertex in $S_1 - S_2$ that is adjacent to x, it must be that either y or z is in $S_1 - S_2$. But then $V(G) - \{x,y,z\} \subseteq S_1 \cap S_2$, implying that $\{x,y,z\} = (S_1 - S_2) \cup (S_2 - S_1)$, contradicting Lemma 2.6.

Finally, suppose $\{x,y,z\} = A(S_1) \cup A(S_2)$. Then $\{x,y,z\} = S_1 \cap S_2$. By Lemma 2.6, there exist $u \in S_1 - S_2$ and $v \in S_2 - S_1$. Define $T_1 = (S_1 - \{u\}) \cup \{v\}$ and $T_2 = (S_2 - \{v\}) \cup \{u\}$. Then $\{T_1, T_2\}$ is a split that crosses S. ∎

Proposition 4.2. *If G is a wheel, then G does not have a good split.*

Proof. Let G be a wheel, and suppose $S=\{S_1,S_2\}$ is a split of G. Let u be a vertex of G to which all other vertices are adjacent.

If $u \in S_1 - S_2$, then, from the definition of a split, u is adjacent to at most one member in $S_2 - S_1$. However, by Lemma 2.6, $|S_2 - S_1| \geq 2$, which implies that u is adjacent to at least two members in $S_2 - S_1$, a contradiction. Thus, $u \in S_1 \cap S_2$.

Note that every vertex of G, other than u, has degree three. This implies that the two S-connections, other than u, are both edges, say e and f. Let $e = wx$ and $f = yz$, where w, $y \in S_1$. Define $T_1 = (S_1 \cup \{z\}) - \{w\}$ and $T_2 = (S_2 \cup \{w\}) - \{z\}$. Then $\{T_1, T_2\}$ is a split that crosses S. ∎

The remainder of this section is devoted to showing that any graph that has a split, but does not have a good split, is either a twirl or a wheel. The basic idea behind proving this is as follows. First, it is shown that two splits of a graph can cross in one of three different ways. Each of the three types of crossing is examined and in each case it is shown constructively that either a good split exists or the graph is either a twirl or a wheel.

The next result provides some properties of crossing splits.

Lemma 4.3. *Let $S = \{S_1, S_2\}$ and $T = \{T_1, T_2\}$ be crossing splits of a minimally 3-connected graph G. Then the following statements hold.*

(a) *$S_1 \cap (T_1 - T_2)$ is nonempty.*

(b) *$|S_1 \cap T_1| \geq 2$.*

(c) *The graph $G[S_1]$ contains at least two T-connections.*

(d) *There exists a vertex of G that is both an S-connection and a T-connection.*

(e) *If $|S_1 \cap T_1| \geq 3$, then $(S_1 \cap T_1) - A(T_1)$ is nonempty.*

Proof. a) If $S_1 \cap (T_1 - T_2)$ is empty, then $S_1 \subseteq (V(G) - (T_1 - T_2)) = T_2$, contradicting the fact that S and T cross.

b) This follows immediately from Lemma 3.2.

c) By part (a), there exist distinct vertices $p \in S_1 \cap (T_1 - T_2)$ and $q \in S_1 \cap (T_2 - T_1)$. By Lemma 2.1, there exist two internally disjoint (p,q)-paths in $G[S_1]$. Evidently, each such path contains a T-connection. Thus, there are two T-connections in $G[S_1]$.

d) By part (c), $G[S_1]$ and $G[S_2]$ each have at least two T-connections. Since there are exactly three T-connections, $G[S_1]$ and $G[S_2]$ share a T-connection. By Lemma 2.4, this T-connection is a vertex, say z. Since $z \in S_1 \cap S_2$, it is also an S-connection.

e) Suppose that $(S_1 \cap T_1) \subseteq A(T_1)$. Since $|S_1 \cap T_1| \geq 3$, it must be that $S_1 \cap T_1 = A(T_1)$.

By part (a), there exist $p \in S_2 \cap (T_2 - T_1)$ and $q \in T_1 \cap (S_2 - S_1)$. Thus, $p \in T_2 - A(T_1)$. Since $q \notin S_1$, it follows that $q \notin A(T_1)$. Thus, $q \notin T_1 - A(T_1)$.

Since G is 3-connected, there exist at least three internally disjoint (p,q)-paths. Moreover, each of these paths contains, as an internal vertex, a vertex of $A(T_1)$. Since $A(T_1) \subseteq S_1$, each such path contains, as an internal vertex, a vertex of S_1. Since $p \in S_2$ and $q \in S_2 - S_1$, each (p,q)-path that contains, as an internal vertex, a vertex of S_1 is either contained in S_2 or contains at least two vertices of $A(S_1)$, at least one of which as an internal vertex. Given that there exist three internally disjoint (p,q)-paths, the latter is impossible, implying that each such path is contained in S_2. But since each such path contains a vertex of S_1, it follows that $A(S_1) = A(S_2) = A(T_1)$.

By applying part (a) once again, $T_1 \cap (S_1 - S_2)$ is nonempty. Now, $T_1 \cap (S_1 - S_2) = T_1 \cap (S_1 - A(S_2)) = (S_1 \cap T_1) - A(S_2) = (S_1 \cap T_1) - A(T_1)$. ∎

The analysis of crossing splits is now broken down into cases, depending on the size of $S_1 \cap T_1$. The next result describes the situation when $|S_1 \cap T_1| = 2$; Figure 3 illustrates the conditions of the lemma.

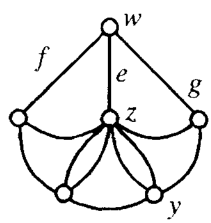

Fig. 3

Lemma 4.4. *Let $S = \{S_1, S_2\}$ and $T = \{T_1, T_2\}$ be crossing splits of a minimally 3-connected graph G. If $S_1 \cap T_1 = \{w, z\}$, then*

(a) *w and z are joined by an edge e,*
(b) *z (say) is an S-connection and a T-connection,*
(c) *w is incident to precisely three edges $\{e,f,g\}$, and*
(d) *f (say) is an S-connection and g is a T-connection.*
(e) *$G\backslash w$ has a 2-separation $\{E_1,E_2\}$ such that $A(G\backslash w,E_1)=\{y,z\}$, where y is not adjacent to w.*

Proof. Let $S_1\cap T_1=\{w,z\}$. By Lemma 4.3(d), there exists a vertex in $S_1\cap T_1$, say z, that is an S-connection and a T-connection. By Lemma 4.3(a), the sets $S_1\cap(T_1-T_2)$ and $T_1\cap(S_1-S_2)$ each contain at least one vertex. Evidently, w is the unique vertex in both of these sets. Since $w\in S_1-S_2$, it is adjacent to at most one vertex of S_2-S_1, and similarly it is adjacent to at most one vertex of T_2-T_1. Since w has degree at least three, it must be that w is adjacent to z, to a vertex of S_2-S_1 via edge f (say), to a vertex of T_2-T_1 via edge g and to no other vertices. Evidently, the edge f is an S-connection and the edge g is a T-connection. Thus, (a)–(d) hold.

Finally, a straightforward analysis shows that $\{E(G\backslash w)-E(G[S_2]),E(G[S_2])\}$ is the desired 2-separation for part (e). ∎

Next is the situation when $|S_1\cap T_1|=3$; Figure 4 illustrates the conditions of the lemma.

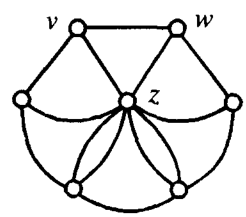

Fig. 4

Lemma 4.5. *Let $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ be crossing splits of a minimally 3-connected graph G. If $S_1\cap T_1=\{v,w,z\}$, then*
(a) *v and w are joined by an edge,*
(b) *v and z are joined by an edge,*
(c) *w and z are joined by an edge,*
(d) *z (say) is an S-connection and a T-connection, and*
(e) *v and w are of degree three.*

Proof. Let $S_1\cap T_1=\{v,w,z\}$. Any vertex of $(S_1\cap T_1)-(A(S_1)\cup A(T_1))$ must be adjacent to at least three vertices of $S_1\cap T_1$, and therefore no such vertex can exist. By 4.3(e), v (say) is a member $A(S_1)-A(T_1)$ and w (say) is a member of $A(T_1)-A(S_1)$. Thus, v is adjacent to at least two members of S_1 and to no members of T_2. It follows that v is adjacent to w and z. Similarly, w is adjacent to v and z. Therefore (a)–(c) hold. Furthermore, by Lemma 4.3(d), the vertex z is an S-connection and a T-connection, implying that (d) holds.

Since z is an S-connection and is adjacent to v, Lemma 2.4 implies that v is not an S-connection. Since $v \in A(S_1)$, it is adjacent to exactly one vertex from S_2. It follows that v has degree three. Similarly, w has degree three. ∎

Lemma 4.6. *Let $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ be crossing splits of a minimally 3-connected graph G. If S is type 1 or 2, $|S_1| \geq 4$ and z is a vertex that is an S-connection and a T-connection, then $G[S_1]$ has a 2-separation $\{E_1,E_2\}$ such that $A(G[S_1],E_1)=\{y,z\}$, where $y \notin A(S_1)$. Moreover, if $\{u,v\}=A(S_1)-\{z\}$, then $u \in V(G[E_1])$ (say) and $v \in V(G[E_2])$.*

Proof. The first step is to show that $G[S_1]$ contains exactly two T-connections. By Lemma 4.3(c), $G[S_1]$ and $G[S_2]$ each contain at least two T-connections. Suppose that $G[S_1]$ contains three T-connections. Since $G[S_1]$ and $G[S_2]$ are edge disjoint, two of the T-connections are vertices in the set $S_1 \cap S_2$. Thus, S is type 1 and the remaining T-connection is either a vertex in $S_1 - S_2$ or an edge that joins two vertices in $S_1 - S_2$. By Lemma 4.3(a), there exist $p \in S_2 \cap (T_1 - T_2)$ and $q \in S_2 \cap (T_2 - T_1)$. Since each (p,q)-path contains a T-connection, the existence of three internally disjoint (p,q)-paths is impossible, contradicting the 3-connectivity of G. Thus, $G[S_1]$ contains exactly two T-connections; by Lemma 4.3(c), at least one is the vertex z.

First, suppose that $|S_1 \cap T_1| = 2$. By Lemma 4.4, there exists a vertex $w \in A(S_1)$ such that w is incident to precisely three edges, $\{e,f,g\}$, with f being an S-connection. It follows that both e and g are edges of $G[S_1]$. Let $E_1=\{e,g\}$ and $E_2=E(G[S_1])-E_1$. Since $|S_1| \geq 4$, it follows that $|E_2| \geq 2$, and so $\{E_1,E_2\}$ is a 2-separation of $G[S_1]$. Moreover, $z \in A(G[S_1],E_1)$ and $w \in V(G[S_1])$. Let y be the other vertex in $A(G[S_1],E_1)$. The result will follow by showing that $y \notin A(S_1)$. To this end choose $p \in (S_1 \cap T_2)-A(S_1)$, which exists by Lemma 4.3(e). (If $|S_1 \cap T_2|=2$, then by Lemma 4.3(d), $|S_1|=3$, a contradiction.) Consider three internally disjoint (p,w)-paths. The (p,w)-path that contains e contains two vertices from $A(S_1)$, namely w and z. The (p,w)-path that contains f also contains two vertices from $A(S_1)$, namely w and the unique vertex of $A(S_1)-\{w,z\}$. Therefore, the (p,w)-path that contains g, and hence y, does not contain any vertex of $A(S_1)$ other than w. Thus, $y \notin A(S_1)$.

Now suppose $|S_1 \cap T_1|, |S_1 \cap T_2| \geq 3$. It is first shown that there exists a vertex y in $A(T_1)-A(S_1)$. By Lemma 4.3(e), there exist $p \in (S_1 \cap T_1)-A(S_1)$ and $q \in (S_1 \cap T_2)-A(S_1)$. If p (say) is a member of $A(T_1)$, then there exists a vertex y $(=p)$ in $A(T_1)-A(S_1)$. If neither p nor q are in $A(T_1)$, the p and q are distinct, implying there exist three internally disjoint (p,q)-paths. Since $G[S_1]$ contains exactly two T-connections, one such path contains two vertices from $A(S_1)$ as internal vertices, and therefore one such path contains no vertices from $A(S_1)$. Moreover, this latter path contains, as an internal vertex, a vertex y from $A(T_1)-A(S_1)$. Since $G[S_1]$ contains exactly two T-connections, it follows that it contains exactly two vertices from $A(T_1)$ (say), which must be y and z. Define E_1 to be the set of edges that have both ends in $S_1 \cap T_1$ and define $E_2=E(G[S_1])-E_1$. Then $\{E_1,E_2\}$ is claimed to be the desired 2-separation. Observe that $A(G[S_1],E_1) \subseteq A(T_1) \cap S_1$, and therefore $|A(G[S_1],E_1)| \leq 2$. Since $(S_1 \cap T_1)-A(S_1)$ and $(S_1 \cap T_2)-A(S_1)$ are nonempty, each of E_1 and E_2 contain at least two edges. Finally, let $\{u,v\}=A(S_1)-\{z\}$; then $u \in V(G[E_1])$ (say) and $v \in V(G[E_2])$ for otherwise G has a 2-separation. ∎

The next results states that if a graph has a particular kind of crossing split, but does not have a good split, then it is a wheel.

Theorem 4.7. *Let G be a minimally 3-connected graph that has a split, but does not have a good split. If G has a pair of crossing splits $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ such that $|S_1 \cap T_1| \leq 3$, then G is a wheel.*

Proof. Let $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ be crossing splits of G such that $|S_1 \cap T_1| \leq 3$. By Lemma 4.3(b), $|S_1 \cap T_1| \geq 2$. By Lemmas 4.4 and 4.5, there exists a vertex z that is an S-connection and a T-connection and a vertex $w \in S_1 \cap T_1$ that has degree three and is adjacent to z.

A set W is constructed as follows. First, set $W \leftarrow \{w\}$. Now repeat the following step until no more vertices can be added to W: If $x \in V(G)-(W \cup \{z\})$ has degree three, and is adjacent to z and to a vertex of W, then set $W \leftarrow W \cup \{x\}$. If $W \cup \{z\} = V(G)$, then G is a wheel. Thus, assume $W \cup \{z\} \neq V(G)$, which implies that $|W \cup \{z\}| \leq |V(G)|-2$.

Define $X_1 = W \cup \{z\}$ and define X_2 to be the vertex set of the graph $G \backslash W$.

First, consider the case when $|W| \geq 2$. Then $X=\{X_1,X_2\}$ is a split of G. Moreover, X is type 3 if z has degree three in G and type 2 otherwise. However, by Lemma 4.3(c), a type 3 split is good, and therefore X is type 2. Since G does not have a good split, there exists a split that crosses X. By Lemma 4.6, the graph $G[X_2]$ has a 2-separation $\{E_1,E_2\}$ such that $A(G[X_2],E_1)=\{y,z\}$, where $y \in X_2 - A(X_2)$.

A similar situation holds when $|W|=1$. By Lemma 4.5, if $|W|=1$, then $W \cup \{z\} = S_1 \cap T_1$. Thus, by Lemma 4.4(e), $G[X_2]$ has a 2-separation $\{E_1,E_2\}$ such that $A(G[X_2],E_1)=\{y,z\}$, where $y \in X_2 - A(X_2)$.

In either case, it follows that $G[X_2]$ has a 2-separation $\{E_1,E_2\}$ such that $A(G[X_2],E_1)=\{y,z\}$, where $y \in X_2 - A(X_2)$. Among all such 2-separations of $G[X_2]$, assume $\{E_1,E_2\}$ is chosen so that $|E_1|$ is minimum. Note that by the construction of W, the set E_1 contains at least three edges. Thus, $\{E_1, E(G)-E_1\}$ is a 3-separation of G. Let $A(G,E_1)=\{u,y,z\}$. By the construction of W, there exists a unique vertex $v \in W$ that is adjacent to u. Also by the minimality of $|E_1|$, the vertex y has degree at least two in $G[E_1]$.

Let $Y=\{Y_1,Y_2\}$ be the split associated with $\{E_1, E(G)-E_1\}$ with Y_1 being the set of vertices having degree greater than one in $G[E_1]$. Note that Y is of type greater than or equal to 1 since the edge uv is a Y-connection. Also, by the construction of W, it follows that $|Y_1|,|Y_2| \geq 4$. The theorem will follow by showing that Y is a good split. To this end, suppose that Y is crossed by some split $Z=\{Z_1,Z_2\}$. By Lemma 4.6, $G[Y_1]$ has a 2-separation $\{F_1,F_2\}$. By Lemma 4.3(d), either y or z is a Y-connection and a Z-connection. If z is a Y-connection and a Z-connection, then a contradiction to the minimality of $|E_1|$ is obtained since $\{F_1, E(G[X_2])-F_1\}$ (say) is a 2-separation of $G[X_2]$ satisfying the properties stated above and $|F_1|<|E_1|$. Thus, y is a Y-connection and a Z-connection. Now by applying Lemma 4.6 to $G[Y_2]$, there exists a 2-separation $\{D_1,D_2\}$ of $G[Y_2]$ such that $v \in V(G[D_1]) - A(G[Y_2],D_1)$ and $z \in V(G[D_2]) - A(G[Y_2],D_1)$. But this contradicts the fact that v and z are adjacent by an edge of $G[Y_2]$. ∎

The remaining part of the section shows that if G does not have a pair of crossing splits as described in Theorem 4.7, then G is a twirl.

Lemma 4.8. *Let G be a minimally 3-connected graph. If $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ are crossing splits of G and $|S_1\cap T_1|\geq 3$, then $|A(S_1\cap T_1)|=3$.*

Proof. Every vertex of $A(S_1\cap T_1)$ is adjacent to a vertex in either S_2-S_1 or T_2-T_1. Therefore $A(S_1\cap T_1)=(S_1\cap T_1)\cap(A(S_1)\cup A(T_1))$.

First, suppose that $|A(S_1\cap T_1)|\leq 2$. Since $|S_1\cap T_1|\geq 3$, the graph $G[S_1\cap T_1]$ has at least two edges. Also, $G[V(G)-(S_1\cap T_1)]$ has at least two edges. Thus, G has a 2-separation, a contradiction.

Now, suppose that $|A(S_1\cap T_1)|>3$. Since $|A(S_1)|=|A(T_1)|=3$, it must be that $A(S_1)-A(T_1)\neq\emptyset$. Let $v\in A(S_1)-A(T_1)$.

Suppose that $A(S_1)\subseteq A(S_1\cap T_1)$. By Lemma 4.3(a), choose $p\in S_1\cap(T_2-T_1)$ and $q\in S_2\cap(T_2-T_1)$. Since $S_1\cap S_2\subseteq A(S_1)\subseteq T_1$, it must be that p and q are distinct. Now every (p,q)-path contains a member of $A(S_1)$. Moreover, since $A(S_1)\subseteq T_1$, each (p,q)-path can only contain a vertex of $A(S_1)$ as an internal vertex. Therefore, among any collection of three internally disjoint (p,q)-paths, at least one such path contains v. Since $A(S_1)\subseteq T_1$, it follows that each (p,q)-path contains a vertex of $A(T_1)$. Since $v\in A(S_1)-A(T_1)$ and $A(S_1)\subseteq A(S_1\cap T_1)$, it follows that $v\in T_1-A(T_1)$. Thus, any (p,q)-path that contains v must also contain at least two members of $A(T_1)$, contradicting the existence of three internally disjoint (p,q)-paths.

Suppose that neither $A(S_1)$ nor $A(T_1)$ is a subset of $A(S_1\cap T_1)$. Thus, $A(S_1\cap T_1)$ consists of exactly two vertices from each of $A(S_1)$ and $A(T_1)$. The vertex of $A(S_1)-A(S_1\cap T_1)$ cannot be identical to the vertex of $A(T_1)-A(S_1\cap T_1)$, for otherwise it is a vertex of $S_1\cap T_1$. It follows that $A(S_1)\cap A(T_1)=\emptyset$, which contradicts Lemma 4.3(d). ∎

Define triads P_1 and P_2 of G to be *coincident* if $G[P_1\cup P_2]$ is the graph $K_{2,3}$.

Lemma 4.9. *Let G be a minimally 3-connected graph that has a split, but does not have a good split. If every pair of crossing splits $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ satisfy $|S_1\cap T_1|\geq 4$, then G has a pair of coincident triads.*

Proof. Let $S=\{S_1,S_2\}$ be a split of G such that $|S_1|$ is as small as possible. Let $T=\{T_1,T_2\}$ be a split that crosses S. By Lemma 4.8, $|A(S_1\cap T_1)|=3$.

Since $|S_1\cap T_1|\geq 4$, the graph $G[S_1\cap T_1]$ has at least three edges, namely three edges incident to a vertex in the set $(S_1\cap T_1)-A(S_1\cap T_1)$. Therefore $\{E(G[S_1\cap T_1]),E(G)-E(G[S_1\cap T_1])\}$ is a 3-separation of G. By the minimality of $|S_1|$, it must be the case that $E(G[S_1\cap T_1])$ is triad of G. Similarly, $E(G[S_1\cap T_2])$ is a triad of G. The proof will be completed by showing that these triads are coincident.

Let u be the degree-three vertex of $G[S_1\cap T_1]$. Then $u\notin A(S_1\cap T_1)$. Now $A(S_1\cap T_1)=(S_1\cap T_1)\cap(A(S_1)\cup A(T_1))$. Suppose that there exists a vertex $v\in S_1\cap(A(T_1)-A(S_1))$. Since $v\in A(T_1)$, it must be adjacent to at least two vertices of T_1, and since $v\notin A(S_1)$, it is adjacent only to members of S_1. Thus, v is adjacent to at least two members of $S_1\cap T_1$. Since $G[S_1\cap T_1]$ is a triad of G, the only vertex of $S_1\cap T_1$ that is adjacent to at least two members of $S_1\cap T_1$ is u. That is, $u=v$. But since $v\in A(T_1)$, it must be adjacent to a vertex of T_2-T_1. This is a contradiction since u is adjacent only to members of T_1. Thus, $A(S_1\cap T_1)=(S_1\cap T_1)\cap A(S_1)$. By Lemma 4.8, $|A(S_1\cap T_1)|=3$, and therefore $A(S_1\cap T_1)=A(S_1)$. That is, u is adjacent to precisely the three vertices of $A(S_1)$. Similarly, the degree-three vertex of $G[S_1\cap T_2]$ is adjacent to precisely the three vertices of $A(S_1)$. It follows that the two triads are coincident. ∎

Theorem 4.10. *Let G be a minimally 3-connected graph that has a split, but does not have a good split. If every pair of crossing splits $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ satisfy $|S_1\cap T_1|\geq 4$, then G is a twirl.*

Proof. By Lemma 4.9, G has a pair of coincident triads, say P_1 and P_2. A set P is constructed as follows. First, set $P\leftarrow P_1\cup P_2$. Now repeat the following step until no more edges can be added to P: If Q is a triad that is disjoint from P and is coincident with P_1, then set $P\leftarrow P\cup Q$. If $P=E(G)$, then G is a twirl. Thus, assume otherwise, and observe that $\{P,E(G)-P\}$ is a 3-separation of G. Define $\{x,y,z\}=A(G,P)$.

Let E_1 be a minimal set of edges contained in $E(G)-P$ such that $A(G,E_1)=\{x,y,z\}$. Define $E_2=E(G)-E_1$. Observe that neither E_1 nor E_2 is a triad of G. Thus, there exists a split $S=\{S_1,S_2\}$ associated with $\{E_1,E_2\}$.

The theorem will follow by showing that S is a good split. To this end, suppose that S is crossed by some split $T=\{T_1,T_2\}$.

The first step is to show that the set of T-connections is given by $A(S_2)=\{x,y,z\}$. By Lemma 4.3(d), the vertex z (say) is a T-connection. For $i\in\{1,2\}$, let p_i be the degree-three vertex of $G[P_i]$. By Lemma 4.3(c), not both of p_1 and p_2 can be members of $A(T_1)$ or $A(T_2)$. Thus, assume $p_1\in T_1-A(T_1)$. By Lemma 4.3(e), there exists a $q\in(S_1\cap T_2)-A(T_2)$. Now, each (p_1,q)-path contains a T-connection. If neither x nor y is a T-connection, then $G[S_2]$ contains only one T-connection, a contradiction to Lemma 4.3(c). Thus, y (say) is a T-connection. Moreover, if x is not a T-connection, then the third T-connection (i.e., the T-connection that is neither y nor z) is not in the graph $G[S_2]$. In that case, choose vertices $r\in S_2\cap(T_1-T_2)$ and $s\in S_2\cap(T_2-T_1)$, which exist by Lemma 4.3(a). Then each (r,s)-path contains a T-connection. Moreover, any (r,s)-path that contains the T-connection that is not in $G[S_2]$ must contain at least two vertices from $A(S_2)$, and therefore at least one vertex from $\{y,z\}$. That is, any such path contains at least two T-connections, contradicting the existence of three internally disjoint (r,s)-paths. Therefore, the set of T-connections if given by $\{x,y,z\}$.

By the minimality of E_1, the graph $G[S_1]\setminus\{x,y,z\}$ is connected. Therefore, $S_1\subseteq T_1$ (say), contradicting the fact that S and T cross. ∎

The proofs of Theorem 1.1 and 1.3 now follow easily.

Proof of Theorem 1.3 Combine Propositions 4.1 and 4.2, Lemma 4.3(b) and Theorems 4.7 and 4.10. ∎

Proof of Theorem 1.1 Combine Theorems 1.2 and 1.3 and Lemma 2.5. ∎

5. A Decomposition Algorithm.

This section briefly outlines an $O(|V(G)|^2|E(G)|)$ algorithm for computing the unique decomposition of a minimally 3-connected graph G that is specified by Theorem 1.1. The algorithm is based on the results of the previous section. For comparison, the unique decomposition of a 2-connected graph discussed in the Introduction can be found in linear time; see Hopcroft and Tarjan [8].

The algorithm runs as follows. Let G be the input graph, and define $D=\{G\}$. At a general iteration, the algorithm selects a graph H from D and attempts to find a good split of H. If successful, then H is replaced in D by the members of the simple decomposition associated with the good split. The process is continued until no member of D has a good split, at which point the unique decomposition has been found.

The main subroutine of the algorithm is a procedure for finding a good split of a graph that is not cyclically 4-connected, a twirl or a wheel. The next result implies that such a procedure need be invoked at most $O(|V(G)|)$ times.

Proposition 5.1. *Let G be a minimally 3-connected graph. Then any decomposition of G has at most* $\max\{1,2|V(G)|-10\}$ *members.*

Proof. The proof is by induction on $|V(G)|$. Let D be a decomposition of G. Let $D_1,\ldots,D_t$ be a sequence of decompositions such that $D_1=\{G\}$, $D_t=D$ and, for $1\le i\le t-1$, D_{i+1} is a simple refinement of D_i. If $t=1$, then the result is evident, and so assume otherwise, and let $D_2=\{G_1,G_2\}$. Then it follows from the definition of the simple decomposition that $|V(G_1)|+|V(G_2)|\le|V(G)|+5$.

It is straightforward to check that any minimally 3-connected graph having at most five vertices is either cyclically 4-connected or a wheel. Therefore, $t\ge 2$ implies that $|V(G)|\ge 6$.

Let D'_i be a decomposition of G_i, for $i\in\{1,2\}$, such that $D=D'_1\cup D'_2$.

First, suppose that $|V(G_1)|,|V(G_2)|\ge 6$. Then by induction, $|D'_i|\le 2|V(G_i)|-10$, for $i\in\{1,2\}$. Now $|D|=|D'_1|+|D'_2|$. Therefore, $|D|\le 2(|V(G_1)|+|V(G_2)|)-20$, which implies that $|D|\le 2|V(G)|-10$, as required.

Second, suppose that $|V(G_1)|\ge 6$ and that $|V(G_2)|\le 5$. Then, by induction $|D'_1|\le 2|V(G_1)|-10$. Also, $|D'_2|=1$. Thus, $|D|\le 2|V(G_1)|-9$. By Lemma 2.6, $|V(G_1)|\le|V(G)|-1$, implying the desired result.

Finally, suppose that $|V(G_1)|,|V(G_2)|\le 5$. Then $|D'_1|=|D'_2|=1$, implying $|D|=2$, as required. ∎

The above result implies that the complexity for finding the decomposition is $|V(G)|$ times the complexity of finding a good split. Given a graph H that is neither a twirl nor a wheel, the procedure outlined below will either find a good split or determine that the graph H is cyclically 4-connected.

The algorithm first works under the assumption that H has a pair of crossing splits. Then Theorems 4.7 and 4.10 imply that H has a good split. Moreover, the proofs of these theorems are constructive and can be turned into algorithm as follows. The first part of the algorithm assumes, as in Theorem 4.7, that G has a pair of crossing splits S and T such that $|S_1\cap T_1|\le 3$.

Choose, if possible, adjacent vertices w and z such that w has degree three and z has degree at least 4. Assuming such a w and z exist, then form the set W as in the proof of Theorem 4.7. Let $X_1=W\cup\{z\}$ and $X_2=V(H\setminus W)$. If $|W|>1$, then $\{X_1,X_2\}$ is a split. If it is of type 3, then, by Lemma 4.3(a), it is a good split. Thus, assume this split is of type 1 or 2. If $H[X_2]$ has a 2-separation of the type described in the proof, then, as in the proof, a good split can be constructed. Determining whether the appropriate type of 2-separation exists in $H[X_2]$ and, if so, constructing the good split can be done in time $O(|E(H)|)$. (See Hopcroft and Tarjan [8] for the 2-separation algorithm.) If $H[X_2]$ does not have a 2-separation of the desired type, then the process is repeated starting with another choice of w or z, or both. If no such w and z exist or if no choice of w and z successfully produces a good split, then H does not have a pair of crossing splits $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ such that $|S_1\cap T_1|\leq 3$. The complexity for this part of the algorithm is $O(|V(H)||E(H)|)$.

Still working under the assumption that H has a pair of crossing splits, the algorithm now assumes, as in Theorem 4.10, that every pair of crossing splits $S=\{S_1,S_2\}$ and $T=\{T_1,T_2\}$ satisfy $|S_1\cap T_1|\geq 4$. First, find a pair of coincident triads, P_1 and P_2 (if they exist). This can be done in $O(|V(H)|)$ time. Let $\{x,y,z\}$ be the vertices common to $H[P_1]$ and $H[P_2]$. If every component of $H\backslash\{x,y,z\}$ is a single vertex, then H is a twirl. Thus, choose a component J of $H\backslash\{x,y,z\}$ having more than one vertex, and let E_1 be the set of edges of J together with those edges of H that have one end in $V(J)$ and the other in $\{x,y,z\}$. Define $E_2=E(H)-E_1$. Then the proof of Theorem 4.10 shows that the split associated with the 3-separation $\{E_1,E_2\}$ is a good split of H.

If the procedures outlined in the previous two paragraphs fail to produce a good split, then it must be the case that H has no pair of crossing splits. That is, every split of H is good, and so it suffices to find any split of H. If H has a split, then it has a cyclic 3-separation. If it has a cyclic 3-separation, then it has a vertex v such that $H\backslash v$ has a 2-separation $\{E_1,E_2\}$ such that neither E_1 nor E_2 is the set of edges incident to a degree-two vertex of $H\backslash v$. Moreover, given any such 2-separation of $H\backslash v$, a cyclic 3-separation of H is easily constructed. Thus, it suffices to find such a 2-separation of $H\backslash v$. This can be done in $O(|E(H)|)$ time using the algorithm of Hopcroft and Tarjan [8]. Considering all possible choices of v yields an $O(|V(H)||E(H)|)$ time algorithm. An alternate method for finding a split of H is to use the algorithm of Kanevsky and Ramachandran [9], which finds all of the 3-separations of H.

Combining the results of the above three paragraphs yields an $O(|V(H)||E(H)|)$ algorithm for finding a good split of a graph H. Therefore there exists an $O(|V(G)|^2|E(G)|)$ algorithm for computing the unique decomposition of the graph G.

A second algorithm for computing the unique decomposition D having the same time bound goes as follows. Consider a decomposition D' of G every member of which is cyclically 4-connected. Then, as in the proof of Theorem 1.2, the decomposition D' is a refinement of D. Moreover, D can be constructed from D' by repeatedly "composing" two members that share a marker vertex and are both twirls (respectively, wheels) provided that the composed graph is also a twirl (respectively, wheel). Finding D' is done as follows. Start with $D'=\{G\}$ and at a general iteration, replace a member H of D' by the members of the simple decomposition associated with an arbitrary split of H, with the process being repeated until no member of D' has a split.

6. Related Issues

Theorems 1.1, 1.2 and 1.3 invite a number of extensions. First, a natural problem to solve is that of characterizing the class of 3-connected graphs that do not have a good split. By Theorem 1.3, this class contains cyclically 4-connected graphs, twirls and wheels. As observed in the Introduction, this containment is proper since a wheel plus an edge joining a pair of non-adjacent degree-three vertices is a 3-connected graph without a good split.

A second problem is to extend Theorems 1.1, 1.2 and 1.3 to matroids. Most of the definitions necessary to state these theorems have matroid analogues; see for example, the work of Rajan [14], Seymour [16] and Truemper [17]. Unique decompositions for 2-connected matroids have been developed by Bixby [1] and Cunningham and Edmonds [4]. It is likely that in a matroid version of Theorems 1.1 and 1.3 the whirl matroids will play a role.

A variation of Theorems 1.1, 1.2 and 1.3 might be obtained by using a different notion of simple decomposition. In particular, suppose that a triangle is used instead of a triad in the construction of the members of the simple decomposition. (This is what Seymour [16] does in his decomposition.) Since triangle and triads are dual, in the sense of matroid duality, dual versions of Theorems 1.1, 1.2 and 1.3 might be possible. In this case minimal 3-connectivity would be defined with respect to contraction rather than deletion, and cyclic 4-connectivity would be replaced by vertical 4-connectivity.

A final related topic is the relationship between the graph decomposition and algorithms for optimization problems defined on the graphs. For the special case of Halin graphs, Cornuéjols, Naddef and Pulleyblank [3] used the decomposition of Theorem 1.1 to develop a polynomial-time algorithm for the traveling-salesman problem. In a subsequent paper, we will extend the Cornuéjols, Naddef and Pulleyblank results to a more general class of graphs and to other optimization problems.

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